Exercise Sheet Solutions #5

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P1. Let $(X, \mathcal{B}, \mu)$ σ-finite measure space. Show that

(a) μ is s-finite.

Solution: As μ is σ-finite, there exist $X_1, X_2, X_3, ... \in \mathcal{B}$ with finite measure such that $X = \bigcup_{n \in \mathbb{N}} X_n$. Notice that we can assume without loss of generality that the sets $(X_n)_{n\in\mathbb{N}}$ are disjoint (otherwise, change X_n by $\tilde{X}_n = X_n \setminus \bigcup_{m=1}^{n-1} X_m$ which has finite measure). Define the measures $\mu_n(A) = \mu(A \cap X_n)$. Then we have that μ_n is a finite measure, and the equation

$$\mu = \sum_{n\in\mathbb{N}} \mu_n$$

follows.

(b) μ is semi-finite.

As μ is σ-finite, there exist $X_1, X_2, X_3, ... \in \mathcal{B}$ with finite measure such that $X = \bigcup_{n\in\mathbb{N}} X_n$. Let $E \subseteq X$ with $\mu(E) = \infty$. By replacing X by E and X_n by $X_n \cap E$, we can assume without loss of generality that $X = E$. If for some $n \in \mathbb{N}$ $\mu(X_n) > 0$ we are done because. If not, then for each $n \in \mathbb{N}$, $\mu(X_n) = 0$ which implies

$$\mu(X) = \mu(\bigcup_{n\in\mathbb{N}} X_n) \leq \sum_{n\in\mathbb{N}} \mu(X_n) = 0,$$

which is a contradiction.

P2. Let $(X, \mathcal{B}, \mu)$ be a measure space. Define $\tilde{\mathcal{B}} = \{E \subseteq X \mid \forall F \in \mathcal{B}, \mu(F) < \infty \Rightarrow E \cap F \in \mathcal{B}\}$.

(a) Prove that $\tilde{\mathcal{B}}$ is a sigma algebra.

Solution: First of all we have $X \in \mathcal{B} \subseteq \tilde{\mathcal{B}}$. Second, for $E \in \tilde{\mathcal{B}}$ we have that for $F \in \mathcal{B}$ with finite measure

$$F \cap E^c = F \cap (F \cap E)^c,$$

and by definition of $\tilde{\mathcal{B}}$, $F \cap E \in \mathcal{B}$, which implies $F \cap E^c \in \mathcal{B}$. Now take $E_1, E_2 \in \tilde{\mathcal{B}}$ and $F \in \mathcal{B}$ with finite measure. Then

$$F \cap (E_1 \cup E_2) = \underbrace{(F \cap E_1)}_{\in\mathcal{B}} \cup \underbrace{(F \cap E_2)}_{\in\mathcal{B}} \in \mathcal{B}.$$

In consequence, $\tilde{\mathcal{B}}$ is a sigma algebra.

(b) Define $\tilde{\mu}$ on M by $\tilde{\mu}(E) = \mu(E)$ for $E \in \mathcal{B}$ and $\tilde{\mu}(E) = \infty$ otherwise. Prove that $\tilde{\mu}$ is a saturated measure on $\tilde{\mathcal{B}}$.

First of all, let us see that $\tilde{\mu}$ is a measure. Clearly $\tilde{\mu}$ is positive and $\tilde{\mu}(\emptyset) = \mu(\emptyset) = 0$. Let $(E_n)_N \subseteq \mathcal{B}$ a pairwise disjoint family. Denote $E = \bigcup_{n\in\mathbb{N}} E_n$. If for some $n \in \mathbb{N}$, $\tilde{\mu}(E_n) \in \tilde{\mathcal{B}} \setminus \mathcal{B}$ then $\mu(E) = \infty$, otherwise as $E_n = E_n \cap E$ we will have that $E_n \in \mathcal{B}$, contradiction. Thus

$$\mu(E) = \infty = \sum_{n\in\mathbb{N}} \mu(E_n).$$

On the other hand, if for each $n \in \mathbb{N}$, $E_n \in \mathcal{B}$ then $E \in \mathcal{B}$ and

$$\tilde{\mu}(E) = \mu(E) = \sum_{n\in\mathbb{N}} \mu(E_n) = \sum_{n\in\mathbb{N}} \tilde{\mu}(E_n).$$

Now we see that $\tilde{\mu}$ is saturated on $\tilde{\mathcal{B}}$. Let $E \in \tilde{\mathcal{B}}$ and $F \in \tilde{\mathcal{B}}$ with $\tilde{\mu}(F) < \infty$. Then, $F \in \mathcal{B}$ by finiteness. Thus, sets with finite measure on $\tilde{\mathcal{B}}$ are precisely sets with finite measure on $\mathcal{B}$. Consequently, by definition of $\tilde{\mathcal{B}}$, we have that $E \in \tilde{\mathcal{B}}$, concluding.

P3. Let $\mathcal{P}$ be a π-system that contains X and $\mathcal{F}$ a family of functions from X to $\mathbb{R}$ such that

- **(a)** $A \in \mathcal{P} \Longrightarrow \mathbb{1}_A \in \mathcal{F}$,
- **(b)** $\mathcal{F}$ is a real vector space: $f, g \in \mathcal{F}$ and $c \in \mathbb{R} \Longrightarrow cf + g \in \mathcal{F}$,
- **(c)** if $(f_n)_{n\in\mathbb{N}}$ is a non-decreasing sequence of positive functions in $\mathcal{F}$ and $f = \lim_{n\to\infty} f_n$ bounded, then $f \in \mathcal{F}$.

Show that $\mathcal{F}$ contains the set $\{f : X \to \mathbb{R} \mid f \text{ is a bounded } \sigma(\mathcal{P})\text{-measurable function}\}$.

Solution: As $X \in \mathcal{P}$, (b), and (c) we have that $\Lambda = \{A : \mathbf{1}_A \in \mathcal{F}\}$ is a λ-system ((b) implies that Λ contains $\emptyset$, is closed under taking complements and under finite disjoint unions, and (c) extends the latter to countable disjoint unions). By (a) implies that $\mathcal{P} \subseteq \Lambda$ and the $\pi - \lambda$ theorem that $\sigma(\mathcal{P}) \subseteq \Lambda$. Statement (b) implies that $\mathcal{F}$ contains all simple functions measurable for $\sigma(\mathcal{P})$, and then (c) implies that $\mathcal{F}$ contains all bounded functions measurable with respect to $\sigma(\mathcal{P})$, concluding.

P4. We will show that if $(X, \mathcal{B}, \mu)$ is a non-atomic probability space, then for all $t \in [0, 1]$, there is $E \in \mathcal{B}$ such that $\mu(E) = t$. For this:

(a) Show that for every $s \in (0, 1)$, there is $E \in \mathcal{B}$ such that $\mu(E) \in (0, s)$.

Solution: By the fact that μ is non-atomic and $\mu(E)$, the we have that there is $F_1 \subseteq E$ such that $\mu(F_1) \in (0, 1)$. Moreover, we can assume that $\mu(F_1) \in (0, 1/2]$, otherwise we replace F_1 by F_1^c. Now, assume that we have constructed F_n such that $\mu(F_n) \in (0, 1/2^n)$ we repeat the previous process to get $F_{n+1} \subseteq F_n$ such that $\mu(F_{n+1}) \in (0, \mu(F_n))$. By possibly replacing F_{n+1} by $F_n \setminus F_{n+1}$, we can assume that $\mu(F_{n+1}) \in (0, \mu(F_n)/2) \subseteq (0, 1/2^{n+1})$. As this holds for each $n \in \mathbb{N}$, we conclude.

(b) Fix $t \in (0, 1)$. Construct a family of disjoint sets $(E_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}$ such that:

- **i)** For each $n \in \mathbb{N}$, $\mu(\bigcup_{i=1}^n E_i) < t$.
- **ii)** If it is possible, for each $n \in \mathbb{N}$, E_n is chosen such that $\mu(E_n) \geq \frac{1}{n}$.

Show that $\mu(\bigcup_{n\in\mathbb{N}} E_n) = t$.

Hint: If the latter is not true, then find $F \in \mathcal{B}$ such that $0 < \mu(F) < t - \mu(\bigcup_{n\in\mathbb{N}} E_n)$. What does this imply for condition **ii)** of the definition of the $(E_n)_{n\in\mathbb{N}}$?

Solution: Using part (a), there is a set E_1' such that $\mu(E_1') < t$. We define E_1 as one of those sets such that $\mu(E_1) \geq 1$ or - if such set does not exist - just as $E_1 = E_1'$. Assume that we have define $(E_i)_{i=1}^n$ with the aforementioned properties. Using part (a), we find $E_{n+1} \subseteq X \setminus \bigcup_{i=1}^n E_i$ such that $0 < \mu(E_{n+1}) < t - \mu(\bigcup_{i=1}^n E_i)$, taking $\mu(E_{n+1}) \geq \frac{1}{n+1}$ if possible. Notice that E_{n+1} is disjoint from $(E_i)_{i=1}^n$ and

$$0 < \mu(\bigcup_{i=1}^{n+1} E_i) = \mu(E_{n+1}) + \mu(\bigcup_{i=1}^{n} E_i) < t, \tag{1}$$

finishing the induction.
Notice that as $\mu(\bigcup_{i=1}^{n} E_i) < t$ for each $n \in \mathbb{N}$, the by continuity of the measure $\mu(\bigcup_{i=1}^{\infty} E_i) \leq t$. If the equality holds, we are done. If not, then $\mu(\bigcup_{i=1}^{\infty} E_i) < t$. By the non-atomicqueness of μ we can find $F \subseteq X \setminus \bigcup_{i=1}^{\infty} E_i$ such that $0 < \mu(F) < t - \mu(\bigcup_{i=1}^{\infty} E_i)$. We fix $N \in \mathbb{N}$ such that $\mu(F) \geq \frac{1}{N}$, and therefore for each $n \geq N$ we have $\mu(F) \geq \frac{1}{n}$. Notice that this implies that $\mu(E_n) \geq \frac{1}{n}$ for each $n \geq N$, given that we could have replaced E_n by $E_n \cup F$. This is impossible due to the fact that

$$\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i) \geq \sum_{i=N}^{\infty} \mu(E_i) \geq \sum_{i=N}^{\infty} \frac{1}{n} = \infty. \tag{2}$$

Hence, $\mu(\bigcup_{i=1}^{\infty} E_i) = t$ is the only possible case.

P5. Verify if the following are examples of π-system and/or λ-systems:

(a) The collection $\mathcal{P} = \{(a, b] : a, b \in \mathbb{R}\}$ of half-open intervals in $\mathbb{R}$.

Solution: This is not a λ system given that $(0, 1] \sqcup (2, 3]$ is not a half-open interval (it is not even connected). It is a π-system though: For $(a, b], (c, d] \in \mathcal{P}$ we have that

$$(a, b] \cap (c, d] = (\max(a, c), \min(b, d)],$$

which is possibly empty (but we are allowing the empty set to be a half-open interval).

(b) Given two measurable spaces $(X, \mathcal{B})$ and $(Y, \mathcal{C})$, the family $\mathcal{P} = \{B \times C : B \in \mathcal{B}, C \in \mathcal{C}\}$ of "rectangles" in $X \times Y$.

Solution: For similar reasons as in part (a), $\mathcal{P}$ is not a λ-system. It is indeed a π-system: Let $B_1, B_2 \in \mathcal{B}$ and $C_1, C_2 \in \mathcal{C}$. Then

$$(B_1 \times C_1) \cap (B_2 \cap C_2) = (B_1 \cap B_2) \times (C_1 \cap C_2) \in \mathcal{P}.$$

(c) For two probability measures μ, ν on a measurable space $(X, \mathcal{B})$, the family $\mathcal{L} = \{E \in \mathcal{B} : \mu(E) = \nu(E)\}$.

Solution: It is not a π-system always (unless $\nu = \mu$). For instance, consider the finite set $[4] = \{1, 2, 3, 4\}$, and the probability measure μ, ν given by

$$\mu(1) = \frac{2}{4}, \mu(2) = \frac{1}{4}, \mu(3) = \frac{1}{4}, \mu(4) = 0;$$

and

$$\nu(1) = \frac{1}{4}, \nu(2) = \frac{2}{4}, \mu(3) = 0, \nu(3) = \frac{1}{4}.$$

Notice that in this case $\{2, 3\}, \{1, 2\} \in \mathcal{L}$ but their intersection is $\{2\}$ which is not in $\mathcal{L}$.
On the other hand, $\mathcal{L}$ is always a λ-system: Clearly $X \in \mathcal{L}$ as $\mu(X) = 1 = \nu(X)$. For $A \in \mathcal{L}$ we have $\mu(A^c) = 1 - \mu(A) = 1 - \nu(A) = \nu(A^c)$ so $\mathcal{L}$ is closed under taking complements. For $A, B \in \mathcal{L}$ disjoint we have that

$$\mu(A \sqcup B) = \mu(A) + \mu(B) = \nu(A) + \nu(B) = \nu(A \sqcup B), \tag{3}$$

concluding that $A \cup B \in \mathcal{L}$. In consequence, $\mathcal{L}$ is a λ-system.